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Program for generating tables of $SU(3)$ coupling coefficients

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Abstract

A C-Language program which tabulates the isoscalar factors and Clebsch-Gordan coefficients for products of representations in $SU(3)$ is presented. These are efficiently computed using recursion relations, and the results are presented in exact precision as square roots of rational numbers. Output is in LaTeX format.

Keywords: $SU(3)$, Clebsch-Gordan coefficients, Wigner coefficients, vector coupling coefficients, isoscalar factors

Classification: 4.2 Computational methods of algebras and groups

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Disclaimer

PROGRAM SUMMARY

Title of program: SU3

Catalogue number:

Program obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue)

Licensing provisions: Persons requesting the program must sign the standard CPC non-profit use license (see license agreement in every issue).

Computers for which the program is designed and tested: DEC VAX 4000/90, DEC Alpha 3000/700, Dell 486, Dell Pentium, HP715/33

Operating systems under which the program has been tested: VMS v6.2, MS-DOS6.x, HP UNIX 9.0

Programming language used: C Language

Memory required to execute: Variable; on the VAX it requires 934 k words for $\mathbf{3} \otimes \mathbf{3}$, 998 k words for $\mathbf{27} \otimes \mathbf{27}$

Disk space used for output: Variable; 2 k words for $\mathbf{3} \otimes \mathbf{3}$, 182 k words for $\mathbf{27} \otimes \mathbf{27}$

No. of bits in a word: 32

No. of processors used: 1

Has the code been vectorized? No

No. of lines in distributed program: 2601

Keywords: $SU(3)$, Clebsch-Gordan coefficients, Wigner coefficients, vector coupling coefficients, isoscalar factors

Classification: 4.2 Computational methods of algebras and groups

Nature of physical problem: Calculations using models based on $SU(3)$ symmetry often require the Clebsch-Gordan coefficients which give the amplitudes for the expansion of direct products of irreducible representations.

Method of solution: Recursion relations [1] are used to construct tables of the isoscalar factors for the product of two irreducible representations. The $SU(2)$ Clebsch-Gordan coefficients [2] [3] are then used with the isoscalar factors to produce the $SU(3)$ Clebsch-Gordan coefficients.

Restrictions on the complexity of the problem: The program is limited by computer memory and maximum size of the integer variables. On the DEC VAX with 32-bit integers, the largest product that has been successfully computed is $\mathbf{27} \otimes \mathbf{27}$.

Typical running time: Running time varies according to the sizes of p_1, q_1, p_2, and q_2 in the product $(p_1, q_1) \otimes (p_2, q_2)$. On the VAX for $\mathbf{3} \otimes \mathbf{3}$ the CPU time is 1.2 sec; for $\mathbf{27} \otimes \mathbf{27}$ the CPU time is 13 min 11 sec. On the Alpha, the latter time is only 1 min 13 sec.

Unusual features of the program: The program has these unique features: it determines the representations in the Clebsch-Gordan series; it uses recursion relations to efficiently compute the isoscalar factors; it presents *exact* results; it provides results as LaTeX [4] formatted tables.

References

[1] H. T. Williams, Symmetry Properties of SU3 Vector Coupling Coefficients, WLUPY-9-93.

[2] E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra (Cambridge University Press, London, 1957).

[3] E. P. Wigner, Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, trans. by J. J. Griffin (Academic Press, New York, 1959).

[4] L. Lamport, LaTeX: A Document Preparation System (Addison-Wesley, Menlo Park CA, 1986).

LONG WRITE-UP

1 Introduction

In calculations in nuclear and particle physics which assume $SU(3)$ symmetry, it is often useful to have values for relevant Clebsch-Gordan coefficients (CGC) *. Some tables of limited extent have been published in the past [1] [5] [6] [7] [8] [9]. Also, programs have been written which generate these coefficients [10] [11] [12] [13]. This new program is meant to replace them. It has some advantages that make it faster and easier to use. Comparative speed is achieved by use of recursion relations to generate the isoscalar factors (ISF), from which the CGC are constructed. This method requires less memory and allows larger tables to be generated. All calculations are done exactly, using integer variables. Output is in the form of LaTeX [14] formatted tables. The code is written in ANSI standard C, so it should run on most conventional platforms.

2 Preliminaries

The Clebsch-Gordan coefficients are the amplitudes for the projection of the product of two irreducible representations ($\mathbf{r}_1$, $\mathbf{r}_2$) of $SU(3)$ onto the separate irreducible representations ($\mathbf{R}_n$) found in the Clebsch-Gordan series:

$$\mathbf{r}_1 \otimes \mathbf{r}_2 = \sum \mathbf{R}_n. \tag{1}$$

They can be defined by

$$|\mathbf{r}_1\alpha_1\rangle\,|\mathbf{r}_2\alpha_2\rangle = \sum \langle \mathbf{R}_i A_i|\mathbf{r}_1\alpha_1\mathbf{r}_2\alpha_2\rangle\,|\mathbf{R}_i A_i\rangle\,, \tag{2}$$

where A_i, α_1, and α_2 denote the quantum numbers specifying the particular states within the representations. Particle physicists are accustomed to the quantum numbers hypercharge y, isospin i, and third component of isospin i_z; herein are also used the "projection" quantum numbers k, l, m, which are related

*The Clebsch-Gordan coefficients are also called *vector coupling coefficients* or *Wigner coefficients*. Readers interested in a good theoretical background are directed to [1] [2] [3] [4].

to y, i, and i_z in a representation (denoted by the representation variables (p, q)) by

$$\begin{aligned} k &= \tfrac{1}{3}(p+2q) + \tfrac{1}{2}y + i, \\ l &= \tfrac{1}{3}(p+2q) + \tfrac{1}{2}y - i, \\ m &= \tfrac{1}{3}(p+2q) + \tfrac{1}{2}y + i_z, \end{aligned} \tag{3}$$

The highest-weight state (shw) in a given representation is defined to be that which has the largest i_z; thus the state with

$$\begin{aligned} k_{\rm shw} = m_{\rm shw} = p + q, \\ l_{\rm shw} = 0. \end{aligned} \tag{4}$$

The (outer) degeneracy of an irreducible representation in the Clebsch-Gordan series (Equation 1) can be determined algebraically from the representation variables [15]. A representation appears in the series if the degeneracy is greater or equal to one, and only in such cases does the program attempt to calculate its coupling coefficients. If we represent the irreducible representations as

$$\begin{aligned} \mathbf{r}_1 &= (p_1, q_1), \\ \mathbf{r}_2 &= (p_2, q_2), \\ \mathbf{R} &= (p, q); \end{aligned} \tag{5}$$

and define

$$\begin{aligned} a &= \tfrac{1}{3}\left[(p_1 - q_1) + (p_2 - q_2) - (p - q)\right], \\ b &= \tfrac{1}{3}\left[(p_1 + 2q_1) + (p_2 + 2q_2) - (p + 2q)\right]; \end{aligned} \tag{6}$$

then if both

1. $a \in \mathbb{Z}$ (integers) (thus $b \in \mathbb{Z}$), and
2. $0 \le b \le \min(q_1 + q_2, p_1 + q_1, p_2 + q_2)$, $-\min(q_1, q_2) \le a \le \min(p_1, p_2)$, $0 \le a + b \le \min(p_1 + q_1, p_2 + q_2, p_1 + p_2)$,

the representation appears in the series and has degeneracy

$$\begin{aligned} d(p, q : p_1, q_1; p_2, q_2) &= 1 \\ &+ \min(q_2, p_1 + q_1, b, p_1 - a) \\ &- \max(0, b - q_1, b - p_2, -a, b - a - q_1, a + b - p_2). \end{aligned} \tag{7}$$

The isoscalar factors F are defined by

$$\begin{aligned}\langle \mathbf{R}\; Y\; I\; I_z | \mathbf{r}_1\; y_1\; i_1\; i_{1z}\; \mathbf{r}_2\; y_2\; i_2\; i_{2z}\rangle &= F(\mathbf{R}, Y, I : \mathbf{r}_1, y_1, i_1; \mathbf{r}_2, y_2, i_2) \\ &\quad \times \langle I\; I_z | i_1\; i_{1z}\; i_2\; i_{2z}\rangle\,,\end{aligned} \tag{8}$$

where the last bracket is an $SU(2)$ CGC, which in the Condon-Shortley phase convention is [16] [17]

$$\begin{aligned}&\langle I\; I_z | i_1\; i_{1z}\; i_2\; i_{2z}\rangle \quad = \\ &\quad \delta(I_z - i_{1z} - i_{2z}) \\ &\quad \times \sqrt{\frac{(I + i_1 - i_2)!(I - i_1 + i_2)!(i_1 + i_2 - I)!(I + I_z)!(2I+1)}{(I + i_1 + i_2 + 1)!(i_1 - i_{1z})!(i_1 + i_{1z})!(i_2 - i_{2z})!(i_2 + i_{2z})!}} \\ &\quad \times \sum_n \frac{(-1)^{n+i_2+i_{2z}}(I + i_2 + i_{1z} - n)!(i_1 - i_{1z} + n)!}{(I - i_1 + i_2 - n)!(I + I_z - n)!n!(i_1 - i_2 - I_z + n)!},\end{aligned} \tag{9}$$

where n runs from

$$n_{\text{low}} = \max(0, i_{1z} - i_1, I_z - i_1 - i_2) \tag{10}$$

to

$$n_{\text{high}} = \min(I + i_2 + i_{1z}, I - i_1 + i_2, I + I_z). \tag{11}$$

Note that

$$\langle I\; I_z | i_1\; i_{1z}\; i_2\; i_{2z}\rangle = (-1)^{I - i_1 - i_2}\,\langle I\; I_z | i_2\; i_{2z}\; i_1\; i_{1z}\rangle\,. \tag{12}$$

2.1 Conventions

The internal phase convention, fixing totally the relative phases between states within a representation, is chosen to be that of de Swart [1]. The irreducible representations of $SU(3)$ can be thought of as consisting of $SU(2)$ multiplets (*isomultiplets*), each at a specific hypercharge. This internal phase convention corresponds to adopting the Condon-Shortley [17] phase convention for the T- and V-spin operators. (In the flavor-$SU(3)$ notation, $T^{\pm}$ interchanges u and d quarks, while $V^{\pm}$ interchanges u and s.)

The overall phase of the representations in the product is also chosen following de Swart [1]. Consider the state of highest weight in the product representation, $|\mathbf{R}\ \text{shw}\rangle$; the state of highest weight in the first factor, $|\mathbf{r}_1\ \text{shw}_1\rangle$; and

the state in the second factor $\mathbf{r}_2 = (p_2, q_2)$ with the highest isospin that couples to shw and shw_1, $|\mathbf{r}_2\ k_{2\max}(\mathrm{shw}, \mathrm{shw}_1)\ l_{2\min}(\mathrm{shw}, \mathrm{shw}_1)\rangle$. The outer phase conventions requires that the Clebsch-Gordan coefficient for coupling these states be real and positive:

$$\langle \mathbf{R}\ \mathrm{shw} | \mathbf{r}_1\ \mathrm{shw}_1\ \mathbf{r}_2\ k_{2\max}(\mathrm{shw}, \mathrm{shw}_1)\ l_{2\min}(\mathrm{shw}, \mathrm{shw}_1)\rangle\ >\ 0. \tag{13}$$

The Condon-Shortley convention for the $SU2$ CGC assures that the corresponding isoscalar factor is also real and positive. The internal and external phase conventions guarantee that all isoscalar factors and Clebsch-Gordan coefficients are real.

In couplings where the degeneracy $d > 1$ there are d distinct sets of CGC, and one must chose by convention a technique for resolving the outer degeneracy. Several distinct techniques for this resolution are described in the literature [18] [19]. The technique chosen here considerably simplifies the determination of isoscalar factors through use of recursion relations, and produces ISF and CGC which share the symmetries under interchange of representations (Racah symmetries), which are familiar from the $SU2$ case. The technique relies on the fact that there are nonvanishing ISFs of the form

$$F(\mathbf{R}, \mathrm{shw} : \mathbf{r}_1, \mathrm{shw}_1; \mathbf{r}_2, k_2, l_2) \tag{14}$$

for at least as many distinct values of k_2 as the degeneracy d. If one chooses ISFs for the $d-1$ highest values of k_2 to be zero, the recursion relations, normalization, and the above-mentioned sign conventions uniquely determine a full set of ISFs. A second set can be determined by choosing the ISFs for the $d-2$ highest values of k_2 to be zero, and insisting that this second set be orthogonal to the first. Subsequent sets of ISFs are formed likewise by successively forcing fewer of these ISFs to zero, and enforcing the condition of orthogonality to those previously determined.

This technique of resolution of the outer degeneracies insures that the coupling coefficients in all cases change by at most a sign under the interchange of $\mathbf{r}_1$ and $\mathbf{r}_2$, in contrast to the "canonical" resolution scheme [19].

2.2 Symmetry Phases

With the conventions of the previous section, all the ISFs and CGCs for $SU(3)$ are fully determined. The resulting symmetries are useful in reducing the number of values which must necessarily be tabulated. Three phases are involved in the symmetries and are defined in [1].

In the case of interchange of the two factor representations,

$$\begin{aligned} F(\mathbf{R}, Y, I : \mathbf{r}_2, y_2, i_2; \mathbf{r}_1, y_1, i_1) &= (-1)^{I-i_1-i_2}\xi_1(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) \\ &\quad F(\mathbf{R}, Y, I : \mathbf{r}_1, y_1, i_1; \mathbf{r}_2, y_2, i_2), \end{aligned} \tag{15}$$

where the factor $(-1)^{I-i_1-i_2}$ comes from Equation 12. The phase $\xi_1(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2)$ depends only on the identity of the representations and on our phase conventions. Consider the reversed product $\mathbf{r}_2 \otimes \mathbf{r}_1$. Suppose that the highest-isospin multiplet in $\mathbf{r}_1$ that couples to the state of highest weight in $\mathbf{r}_2$ and the shw of $\mathbf{R}$ with nonzero ISF has quantum numbers y_1' and i_1'. Then

$$\xi_1(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) = (-1)^{I_\mathrm{h}-i_1'-i_{2\mathrm{h}}} \times \mathrm{sign}\left[F(\mathbf{R}, \mathrm{shw} : \mathbf{r}_1, y_1', i_1'; \mathbf{r}_2, \mathrm{shw}_2)\right], \tag{16}$$

where $\mathrm{sign}(x) = x/|x|$.

Should the second factor and the product representations be interchanged, a multiplicative phase and a term related to the ratio of the representations' dimensions is generated:

$$\begin{aligned} &F(\overline{\mathbf{r}}_2, \text{-}y_2, i_2 : \mathbf{r}_1, y_1, i_1; \overline{\mathbf{R}}, \text{-}Y, I) \quad = \\ &\quad (-1)^{i_1+y_1/2}\xi_2(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) \\ &\quad \times\sqrt{\frac{(p_2+1)(q_2+1)(p_2+q_2+2)(2I+1)}{(p+1)(q+1)(p+q+2)(2i_2+1)}} \\ &\quad \times F(\mathbf{R}, Y, I : \mathbf{r}_1, y_1, i_1; \mathbf{r}_2, y_2, i_2). \end{aligned} \tag{17}$$

If all the representations are conjugated, then a phase ξ_3 enters:

$$\begin{aligned} F(\overline{\mathbf{R}}, Y, I : \overline{\mathbf{r}}_1, y_1, i_1; \overline{\mathbf{r}}_2, y_2, i_2) &= (-1)^{I-i_1-i_2}\xi_3(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) \\ &\quad \times F(\mathbf{R}, \text{-}Y, I : \mathbf{r}_1, \text{-}y_1, i_1; \mathbf{r}_2, \text{-}y_2, i_2). \end{aligned} \tag{18}$$

Consider the isomultiplets described in Section 2.1. Then from Equation 12 we find that

$$\xi_3(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) = (-1)^{I_\mathrm{h}-i_{1\mathrm{h}}-i_2'}. \tag{19}$$

3 Calculational Method

3.1 Recursion Relations for Isoscalar Factors

Recursion relations among the CGC for $SU(3)$ can be generated by applying one of the $SU(3)$ ladder operators to a pair of $SU(3)$ single particle states coupled (via CGCs) to a state of good $SU(3)$ quantum numbers. Using the linearity of the ladder operators [1] (e.g. $K_+ = K_{1+} + K_{2+}$, where K_+ is the ladder operator for the coupled state, and K_{1+} and K_{2+} are the corresponding operators for the 1 and 2 states), and orthogonality of states with different $SU(3)$ quantum numbers, one can find four-term recursion relations for the CGCs. Making use of explicit analytic forms for the SU2 Clebsch-Gordan coefficients these can be turned into four-term recursion relations for the $SU(3)$ isoscalar factors. Two of the ladder operators (I_+ and I_-) generate recursions only within the SU2 variables and produce only identities for the ISF. From the remaining four ladder operators expressions fully sufficient to generate all the $SU(3)$ ISFs have been derived.

The simplest language for presenting the recursion relations is to use (p, q) to indicate an irreducible representation, and (k, l, m) to specify a state within the representation. The scheme involves, as a first step, using one of the following relations to establish values for all the isoscalar factors representing coupling to the state of highest weight (i.e. $k = m = p + q, l = 0$). A simplified notation which omits the p's and q's, and which uses the symbol shw to represent $k = p+q$, $l = 0$ will be adopted. Thus, for example,

$$F(p, q, k = p + q, l = 0 : p_1, q_1, k_1, l_1; p_2, q_2, k_2, l_2) = F(\text{shw} : k_1, l_1; k_2, l_2). \quad (20)$$

The two recursion relations involving the states of highest weight are

$$\begin{aligned} 0 \;=\; & a_1 F(\text{shw} : k_1 - 1, l_1; k_2, l_2) + a_2 F(\text{shw} : k_1, l_1; k_2 - 1, l_2) \\ & + a_3 F(\text{shw} : k_1, l_1 - 1; k_2, l_2) + a_4 F(\text{shw} : k_1, l_1; k_2, l_2 - 1). \end{aligned} \quad (21)$$

where

$$\begin{aligned} a_1 \;=\; & \sqrt{(k_1 + 1)(k_1 - q_1)(p_1 + q_1 - k_1 + 1)(p + q + k_1 - l_1 + k_2 - l_2 + 3)} \\ & \times \sqrt{\frac{2(p + q + k_1 - l_1 - k_2 + l_2 + 1)}{(k_1 - l_1)(k_1 - l_1 + 1)}}, \end{aligned}$$

$$
\begin{aligned}
a_2 &= a_1(1 \leftrightarrow 2), \\
a_3 &= -\sqrt{l_1(q_1 - l_1 + 1)(p_1 + q_1 - l_1 + 2)} \\
&\quad \times \sqrt{(-p - q + k_1 - l_1 + k_2 - l_2 + 1)} \\
&\quad \times \sqrt{\frac{2(p + q - k_1 + l_1 + k_2 - l_2 + 1)}{(k_1 - l_1 + 1)(k_1 - l_1 + 2)}}, \\
a_4 &= -a_3(1 \leftrightarrow 2);
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
0 &= b_1 F(\mathrm{shw} : k_1 + 1, l_1; k_2, l_2) + b_2 F(\mathrm{shw} : k_1, l_1; k_2 + 1, l_2) \\
&\quad + b_3 F(\mathrm{shw} : k_1, l_1 + 1; k_2, l_2) + b_4 F(\mathrm{shw} : k_1, l_1; k_2, l_2 + 1).
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
b_1 &= \sqrt{(k_1 + 2)(k_1 - q_1 + 1)(p_1 + q_1 - k_1)(-p - q + k_1 - l_1 + k_2 - l_2 + 1)} \\
&\quad \times \sqrt{\frac{2(p + q - k_1 + l_1 + k_2 - l_2 + 1)}{(k_1 - l_1 + 1)(k_1 - l_1 + 2)}}, \\
b_2 &= -b_1(1 \leftrightarrow 2), \\
b_3 &= \sqrt{(l_1 + 1)(q_1 - l_1)(p_1 + q_1 - l_1 + 1)(p + q + k_1 - l_1 + k_2 - l_2 + 3)} \\
&\quad \times \sqrt{\frac{2(p + q + k_2 - l_2 - k_2 + l_2 + 1)}{(k_1 - l_1)(k_1 - l_1 + 1)}}, \\
b_4 &= b_3(1 \leftrightarrow 2).
\end{aligned}
\tag{24}
$$

Once the states of highest weight have been determined, two other relations are sufficient to step from these to any non-shw state. They are

$$
\begin{aligned}
F(k, l : k_1, l_1; k_2, l_2) = {} & c_1 F(k + 1, l - 1 : k_1, l_1; k_2, l_2) \\
& + c_2 F(k, l - 1 : k_1, l_1 - 1; k_2, l_2) \\
& + c_3 F(k, l - 1 : k_1, l_1; k_2 - 1, l_2) \\
& + c_4 F(k, l - 1 : k_1, l_1; k_2, l_2 - 1),
\end{aligned}
\tag{25}
$$

where

$$
c_1 = \alpha \sqrt{\frac{(k + 2)(k - q + 1)(p + q - k)(k_1 - l_1 + k_2 - l_2 - k + l)}{(k - l + 2)^2 (k - l + k_1 - l_1 + k_2 - l_2 + 4)}},
$$

$$
\begin{aligned}
c_2 &= 2\alpha\sqrt{\frac{l_1(q_1-l_1+1)(p_1+q_1-l_1+2)}{(k_1-l_1+2)(k-l+k_1-l_1+k_2-l_2+4)}}, \\
&\quad \times\sqrt{\frac{(k_1-l_1+1)}{(k-l-k_1+l_1+k_2-l_2+2)}} \\
c_3 &= -\alpha\sqrt{\frac{(k_2+1)(k_2-q_2)(k_1-l_1+k_2-l_2-k+l)}{(k_2-l_2)(k_2-l_2+1)}} \\
&\quad \times\sqrt{\frac{(p_2+q_2-k_2+1)}{(k-l-k_1+l_1+k_2-l_2+2)}}, \\
c_4 &= \alpha\sqrt{\frac{l_2(q_2-l_2+1)(k-l+k_1-l_1-k_2+l_2+2)}{(k_2-l_2+1)(k_2-l_2+2)}} \\
&\quad \times\sqrt{\frac{(p_2+q_2-l_2+2)}{(k-l+k_1-l_1+k_2-l_2+4)}};
\end{aligned}
\tag{26}
$$

(27)

and

$$
\alpha=\sqrt{\frac{(k-l+2)^2(k-l-k_1+l_1+k_2-l_2+2)}{l(q-l+1)(p+q-l+2)(k-l+k_1-l_1-k_2+l_2+2)}}; \tag{28}
$$

and

$$
\begin{aligned}
F(k,0:k_1,l_1;k_2,l_2) &= d_1F(k+1,0:k_1+1,l_1;k_2,l_2) \\
&\quad +d_2F(k+1,0:k_1,l_1;k_2+1,l_2) \\
&\quad +d_3F(k+1,0:k_1,l_1;k_2,l_2+1),
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
d_1 &= 2\beta\sqrt{\frac{(k_1+2)(k_1-q_1+1)}{(k_1-l_1+2)(k-l+k_1-l_1+k_2-l_2+4)}}, \\
&\quad \times\sqrt{\frac{(p_1+q_1-k_1)(k_1-l_1+1)}{(k-l+k_1-l_1-k_2+l_2+2)}} \\
d_2 &= \beta\sqrt{\frac{(k_2+2)(k_2-q_2+1)(p_2+q_2-k_2)}{(k_2-l_2+1)(k_2-l_2+2)}}
\end{aligned}
\tag{30}
$$

$$\times\sqrt{\frac{(k-l-k_1+l_1+k_2-l_2+2)}{(k-l+k_1-l_1+k_2-l_2+4)}},$$

$$d_3 = \beta\sqrt{\frac{(l_2+1)(q_2-l_2)(p_2+q_2-l_2+1)}{(k_2-l_2)(k_2-l_2+1)}}$$

$$\times\sqrt{\frac{(k_1-l_1+k_2-l_2-k+l)}{(k-l+k_1-l_1-k_2+l_2+2)}}$$

and

$$\beta = \sqrt{\frac{(k+2)}{(k-q+1)(p+q-k)}}. \tag{31}$$

The $SU(3)$ CGC are found by using Equation 8. The corresponding $SU(2)$ coefficients are determined through Equation 10.

3.2 Algorithm for Isoscalar Factors

The use of the above recursion relations to generate isoscalar factors is far from trivial, and a detailed description of the algorithm is given in [20]. A summary of that procedure is given here.

The ISFs for coupling to a state of highest weight can be thought of as occupying lattice sites in an irregularly-shaped volume in a three-dimensional space. With the restriction provided by hypercharge conservation, these ISFs can be thought of as functions of $s \equiv I_1 + I_2$, k_1, and k_2. The two recursion relations above for the shw states each have two terms with the same s value, and two others with s values smaller by one $s-1$. All ISFs with the maximum allowed s value for a particular coupling are related, therefore, by a two-term recursion relation. Setting one of these equal to one (to be fixed via normalization later), all others are simply constructed. Because of the shaped of the lattice, one can in most cases find a particular single ISF with $s = s_{\max} - 1$ which can be determined by those with $s = s_{\max}$: once it is determined, all others with the same s value can be evaluated through the complete four-term recursion relations based on values already known. Repeating this logic, one moves to a single ISF at the next lowest s value from which all others with the same s can be determined, continuing until all ISFs are known. Normalization requires the

sum of the squares of all the ISFs so determined must be 1, so each is multiplied by the appropriate normalization factor.

When dealing with ISFs of degeneracy greater than one, it is necessary to make several choices of ISF values in order for the algorithm to succeed. Which are chosen, and what values they are given, determines the choice of degeneracy resolution. The criteria mentioned in an earlier section is equivalent to choosing ISFs for the $d-1$ highest values of s to be zero, and following the algorithm of the previous paragraph beginning with the $s = s_{\max} - d + 1$ values. A second set begins with the $s = s_{\max} - d + 2$values, and the further condition of orthogonality of these ISFs to the previous ones. Subsequent sets of ISFs are formed likewise by successively starting with higher values of s, and enforcing the condition of orthogonality to those previously determined.

Regardless of degeneracy, there are particular combinations of representations whose ISFs cannot be determined as above due to a failure in the ability to step to lower values of s. In every such case, the algorithm succeeds for the conjugated ISFs, and the exchange symmetry relation of Equation 17 can be used to find the original values.

4 Program Structure

We define a variable type `fraction` as an array of two long integers, n and m, to represent a number of the form $\sqrt{n/m}$. All $SU(3)$ CGCs and ISFs can be exactly represented this way. The routines that manipulate variables of the type `fraction` are

`reducefrac()`	Reduces a fraction to its lowest form.
`setequalfrac()`	Assigns one fraction to be equal to another.
`addfrac()`	Adds two fractions.
`subfrac()`	Subtracts two fractions.
`multfrac()`	Multiplies two fractions.
`imultfrac()`	Multiplies a fraction by a long integer.
`divfrac()`	Divides two fractions.

`absfrac()`	Changes a fraction to its absolute value and returns its sign.
`factorialfrac()`	Finds the factorial of a fraction, if it is integral.
`equalfrac()`	Returns true if and only if two fractions are equal.
`comparefrac()`	Returns true if and only if the first fraction is greater than the second.
`addirrat()`	Adds two numbers of the form $\sqrt{n/m}$. Since the coefficients of $SU(3)$ always take this form, no incompatibilities are expected. An error message is written to the log file if it is not possible to find $$\sqrt{\frac{N}{M}} = \sqrt{\frac{n_1}{m_1}} + \sqrt{\frac{n_2}{m_2}}. \quad (32)$$

Global variables are

`p1, q1, p2, q2`	The identities of the factor representations (p_1, q_1) and (p_2, q_2).
`numprod`	The number of product representations in the Clebsch-Gordan series.
`p[], q[]`	The identities of the product representations (p_i, q_i).
`names[]`	The names of the representations in the product.
`texnames[]`	The LaTeX names of the representations in the product.
`ISFtable[][][][][][]`	Array which holds the isoscalar factors. Its elements are of the type `fraction`.
`SHWtable[][][][]`	Array which holds the isoscalar factors for the states of highest weight. Its elements are of the type `fraction`.
`step_down[][]`	Array which holds certain ratios of ISFs for `doshw()`.

`flag`	An integer flag used by `doslice()` to control `doshw()`.
`errcount`	Cumulative count of errors in the program's execution.
`nameconv, labelconv`	Integers denoting the user's choices for labelling conventions (see below).
`maxr, mink1, maxk1, maxl1, mink2, maxk2, maxl2`	Limits used in dimensioning the various arrays.

Other subordinate routines included are

`sizerep()`	Actually a macro which returns the dimension of a representation.
`ipower()`	Raises a long integer to a power.
`factorial()`	Returns the factorial of a long integer.
`gcd()`	Returns the greatest common devisor of two long integers. It is employed by `reducefrac()`.
`isqrt()`	Approximates the square root of a long integer with another long integer.
`yii()`	Gives the hypercharge and isospin of a state with given k, l, m.
`digitstring()`	Gives a character representing a one-digit integer. Used by `numstring`.
`numstring()`	Returns a character string that represents an integer. Used by `namerep()`.
`namerep()`	Gives the text name and LaTeX name of a representation. Naming conventions are explained below.
`shw()`	Gives the hypercharge and isospin of the highest-weight state of a representation.
`initlims()`	Sets the values of `mink1`, `maxk1`, `maxl1`, `mink2`, `maxk2`, `maxl2`, and `maxr`.

`inittable()`	Allocates memory space for `ISFtable` and initializes its values to zero.
`kill_table()`	Releases the memory used by `ISFtable`.
`putentry()`	Places an entry into `ISFtable`. Because of hypercharge conservation, one index is redundant for `ISFtable`; this is used by `putentry()` and `getentry()` to reduce the size of the array for `ISFtable`.
`getentry()`	Checks to see if indices defining an ISF are all in range, and if so, returns an entry from `ISFtable`.
`initshw()`	Allocates memory space for `SHWtable` and initializes its values to zero.
`killshw()`	Releases the memory used by `SHWtable`.
`putentrySHW()`	Places an entry into `SHWtable`.
`getentrySHW()`	Returns an entry from `SHWtable`.
`printfraction()`	Writes a fraction in LaTeX format to one of the output files.
`printline()`	Writes a line to both `cle.tex` and `iso.tex`.
`numstatesi()`	Counts the isomultiplets of given hypercharge and isospin in the product.
`numstatesc()`	Counts the states of given hypercharge and third component of isospin in the product.
`getconvs()`	Prompts for and inputs the naming and labelling conventions (see below) from the user.

The central routines are

`su2clebsch()`	Calculates an $SU(2)$ Clebsch-Gordan coefficient with Equation 10.
`degeneracy()`	Returns the number of times a representation appears in the Clebsch-Gordan series.

`series()`	Finds the representations in the Clebsch-Gordan series. Also sets the value of `numprod`.
`texphases()`	Tabulates the symmetry phases ξ_1 and ξ_3 into `iso.tex`.
`normhor()`	Checks that each row (for each isospin) of the ISF table has unit norm.
`normvert()`	Checks that each column of the ISF table has unit norm.
`checktable()`	Calls `normhor()` and `normvert()`.
`checksym()`	Checks that the shw ISFs are symmetric under the exchange of $\mathbf{r}_1$ and $\mathbf{r}_2$, in the cases where $p1 = p2$, $q1 = q2$. It compares only the absolute values of the ISFs. It is called by `doslice()`.
`textable()`	Loops over the quantum numbers of the factors and products and tabulates the isoscalar factors into `iso.tex`. Calls `su2clebsch` and tabulates the Clebsch-Gordan coefficients into `cle.tex`.
`coef`x`()`	Returns a coefficient of the recursion relations in Equations 21-31. Here x is one of `a1`, `a2`, `a3`, `a4`, `b1`, `b2`, `b3`, `b4`, `c1`, `c2`, `c3`, `c4`, `d1`, `d2`, `d3`, `alpha`, `beta`.
`l1step()`	Steps up or down in l_1 for `doshw()`.
`k1step()`	Steps up or down in k_1 for `doshw()`.
`doshw()`	Fills the parts of `ISFtable` for the highest-weight states of the product representations.
`doslice()`	Fills `ISFtable` for a given (p_i, q_i) in the product. Uses the algorithm given in the previous section. Calls `doshw()` to handle the states of highest weight. Calls `checksym()` for cases of $p1 = p2$, $q1 = q2$.

`main()`	Prompts for p_1, q_1, p_2, q_2. Calls `getconvs()` to input labelling and naming conventions. Uses `degeneracy` to determine the identities of the product representations and calls `doslice()` for each. Calls `textable()` to generate output. Also calls `normhor()` and `normvert()`.

5 Using the Program

5.1 User Input

The user, upon initiating the code, is queried to supply the identities of the factor representations in the form (p_1, q_1), (p_2, q_2), where each is an integer. The naming and labelling conventions must also be chosen. The choices for naming convention are

1. Representations are named as in [21], which is popular among particle physicists. Representations are named by their dimension and bars are added based on triality

$$t = (p - q) \bmod 3. \tag{33}$$

   Representations with $t = 1$ are unbarred, and those with $t = 2$ are barred. One exception is that $\mathbf{6} \equiv (2, 0)$. Representations with $t = 0$ have bars if and only if $q > p$.

2. Representations are named by their dimension and bars are added if and only if $q > p$.

3. Representations are named by (p, q).

In choices 1 and 2 above, there is an ambiguity when two representations with different (p, q) have the same dimension. We distinguish them using primes. In

the naming scheme 1 the lowest-lying representations with same dimensions are

$$\begin{array}{lcllcllcl}
\mathbf{15} & = & (2,1), & \mathbf{15'} & = & (4,0), & & & \\
\mathbf{105} & = & (6,2), & \mathbf{105'} & = & (13,0), & & & \\
\mathbf{120} & = & (3,5), & \mathbf{120'} & = & (1,9), & \mathbf{120''} & = & (0,14), \\
\mathbf{195} & = & (9,2), & \mathbf{195'} & = & (1,12), & & & \\
\mathbf{210} & = & (4,6), & \mathbf{210'} & = & (19,0), & & & \\
\mathbf{231} & = & (2,10), & \mathbf{231'} & = & (0,20). & & &
\end{array} \tag{34}$$

The choices for labelling are

1. States are labelled by hypercharge and isospin y, i, i_z. Isomultiplets are labelled by y and i.

2. States are labelled by projection quantum numbers k, l, m. Isomultiplets are labelled by k and l.

5.2 Error Messages

The program generates three files during its execution. The log file (`logfile` = `su3.log`) contains messages and reports the progress of the program. See Sample 1 below. Any error messages are written to the logfile. The most important errors are also written to the console. They are:

`WARNING: insufficient memory`
indicates that the free memory in the computer is too small to allow dimensioning of the needed arrays.

`WARNING: integer overflow`
indicates that the integer mathematics has generated numbers larger than the "long" integers defined by the computer.

`WARNING: Table is not horizontally normal`
indicates that the table of ISFs has nonnormal rows. A row is given by $k1, l1, k2, l2$, and the isospin of the states in the product.

`WARNING: Table is not vertically normal`
indicates that the table of ISFs has nonnormal columns. A column is given by $\mathbf{R}, k, l$.

In addition, for cases in which $p1 = p2$ and $q1 = q2$, a quick check is made of the exchange symmetry. If a nonsymmetric entry is found, the message

`WARNING: n=4 SHW ISFs are not symmetric`

(for example) appears.

These errors indicate that a serious problem has occurred. Other error messages are written only to `logfile` and are self-explanatory.

5.3 Output

The ISFs are written in LaTeX format to the file `isofile` = `iso.tex`. In it, representations are named according to the naming choice made at the beginning of execution, and states are labelled according to the labelling choice. Multiply degenerate representations are distinguished by subscripts. A square root is assumed to appear over the unsigned part of each entry. Thus,

$\mathbf{r}_1 \otimes \mathbf{r}_2$				$\mathbf{R}$
				Y
				I
y_1	i_1	y_2	i_2	$\pm C$

(35)

means that the isoscalar factor

$$F(\mathbf{R}, Y, I : \mathbf{r}_1, y_1, i_1; \mathbf{r}_2, y_2, i_2) = \pm\sqrt{C} \tag{36}$$

in the (y, i) labelling convention. In the (k, l) convention, this would appear as

$\mathbf{r}_1 \otimes \mathbf{r}_2$				$\mathbf{R}$
				k
				l
k_1	l_1	k_2	l_2	$\pm C$

(37)

The symmetry factors ξ_1 and ξ_2 are also given in this file. For example,

$\mathbf{r}_1 \otimes \mathbf{r}_2$	$\mathbf{R}$
ξ_1	$+$
ξ_2	$-$

(38)

means that

$$\begin{aligned} \xi_1(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) &= +1, \\ \xi_2(\mathbf{R} : \mathbf{r}_1; \mathbf{r}_2) &= -1. \end{aligned} \tag{39}$$

The Clebsch-Gordan coefficients are written in LaTeX format to the file `clefile` = `cle.tex`. Again, a square root is assumed over the unsigned part of each entry. In the (y, i, i_z) labelling convention, the format is

$\mathbf{r}_1 \otimes \mathbf{r}_2$						$\mathbf{R}$ Y I I_z
y_1	i_1	i_{1z}	y_2	i_2	i_{2z}	$\pm C$

(40)

for

$$\langle \mathbf{R}\ Y\ I\ I_z | \mathbf{r}_1\ y_1\ i_1\ i_{1z}\ \mathbf{r}_2\ y_2\ i_2\ i_{2z} \rangle = \pm\sqrt{C}. \tag{41}$$

In the (k, l, m) notation this is

$\mathbf{r}_1 \otimes \mathbf{r}_2$						$\mathbf{R}$ k l m
k_1	l_1	m_1	k_2	l_2	m_2	$\pm C$

(42)

Samples 2 and 3 below show examples of `iso.tex` and `cle.tex` after processing by LaTeX.

6 Sample Output

Presented here are sample output files from a successful run. The product computed is **15** $\otimes$ **8** = (2, 1) $\otimes$ (1, 1), which has a doubly degenerate **15**. In this run, we have chosen the naming convention of [21] and the labelling convention of hypercharge and isospin (y, i). Sample 1 is the log file `su3.log`. It echoes our naming and labelling conventions, informs us of the program's progress, and tells us that the table of isoscalar factors is properly normalized. Sample 2 is the LaTeX file containing the isoscalar factors. The Clebsch-Gordan coefficients are in the LaTeX file of Sample 3. Notation for the tables is explained in the previous section.

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SAMPLE CAPTIONS

Sample 1: Log file (`su3.log`) from a sample run calculating $\mathbf{15} \otimes \mathbf{8} = (2,1) \otimes (1,1)$..

Sample 2: Output file (`iso.tex`) from a run calculating $\mathbf{15} \otimes \mathbf{8}$. These are the isoscalar factors. Notation is explained in Section 4. The output has already been processed by LaTeX. Only the first page is shown, in order to save space.

Sample 3: Output file (`cle.tex`) from a run calculating $\mathbf{15} \otimes \mathbf{8}$. These are the Clebsch-Gordan coefficients. Notation is explained in Section 4. The output has already been processed by LaTeX. Only the first page of the file is shown, in order to save space.

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```
SU3:  beginning

getconvs:  naming convention is of Slansky
getconvs:  labelling convention is (y, i, i3)

SU3:  calculating 15 x 8

SU3:  working on 3 with degeneracy = 1
SU3:  working on 6* with degeneracy = 1
SU3:  working on 15' with degeneracy = 1
SU3:  working on 15 with degeneracy = 2
SU3:  working on 24 with degeneracy = 1
SU3:  working on 42 with degeneracy = 1

checktable:  table is horizontally normal
checktable:  table is vertically normal

SU3:  writing output files

SU3:  ending
```

Isoscalar Factors for $\mathbf{15} \otimes \mathbf{8}$

$\mathbf{15} \otimes \mathbf{8}$	$\mathbf{3}$	$\overline{\mathbf{6}}$	$\mathbf{15'}$	$\mathbf{15}_1$	$\mathbf{15}_2$	$\mathbf{24}$	$\mathbf{42}$
ξ_1	+	+	−	−	+	−	+
ξ_3	+	−	+	−	+	+	+

				$\mathbf{24}$
				$\frac{7}{3}$
				$\frac{1}{2}$
$\frac{4}{3}$	1	1	$\frac{1}{2}$	-1

				$\mathbf{42}$
				$\frac{7}{3}$
				$\frac{3}{2}$
$\frac{4}{3}$	1	1	$\frac{1}{2}$	1

				$\overline{\mathbf{6}}$	$\mathbf{24}$
				$\frac{4}{3}$	$\frac{4}{3}$
				0	0
$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$-\frac{4}{5}$
$\frac{4}{3}$	1	0	1	$-\frac{4}{5}$	$-\frac{1}{5}$

				$\mathbf{15}_1$	$\mathbf{15}_2$	$\mathbf{24}$	$\mathbf{42}$
				$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
				1	1	1	1
$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{400}{1647}$	$-\frac{1}{61}$	$-\frac{4}{27}$	$\frac{16}{27}$
$\frac{1}{3}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{529}{3294}$	$-\frac{32}{61}$	$-\frac{8}{27}$	$-\frac{1}{54}$
$\frac{4}{3}$	1	0	0	$\frac{49}{183}$	$-\frac{4}{61}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{4}{3}$	1	0	1	$\frac{361}{1098}$	$\frac{24}{61}$	$-\frac{2}{9}$	$\frac{1}{18}$

				$\mathbf{15'}$	$\mathbf{42}$
				$\frac{4}{3}$	$\frac{4}{3}$
				2	2
$\frac{1}{3}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{4}{3}$	1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$

				$\mathbf{3}$	$\overline{\mathbf{6}}$	$\mathbf{15}_1$	$\mathbf{15}_2$	$\mathbf{24}$	$\mathbf{42}$
				$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
				$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{2}{3}$	0	1	$\frac{1}{2}$	$\frac{1}{40}$	$\frac{3}{20}$	$-\frac{50}{183}$	$-\frac{9}{488}$	$-\frac{4}{15}$	$\frac{4}{15}$
$-\frac{2}{3}$	1	1	$\frac{1}{2}$	$-\frac{3}{20}$	$\frac{1}{10}$	$\frac{121}{549}$	$-\frac{75}{244}$	$-\frac{8}{45}$	$-\frac{2}{45}$
$\frac{1}{3}$	$\frac{1}{2}$	0	0	$-\frac{9}{80}$	$-\frac{3}{40}$	$\frac{4}{183}$	$-\frac{121}{976}$	$\frac{2}{15}$	$\frac{8}{15}$
$\frac{1}{3}$	$\frac{1}{2}$	0	1	$-\frac{1}{80}$	$-\frac{5}{24}$	$\frac{196}{1647}$	$\frac{225}{976}$	$-\frac{10}{27}$	$\frac{8}{135}$
$\frac{1}{3}$	$\frac{3}{2}$	0	1	$\frac{2}{5}$	$-\frac{4}{15}$	$\frac{2}{1647}$	$-\frac{18}{61}$	$-\frac{4}{135}$	$-\frac{1}{135}$
$\frac{4}{3}$	1	-1	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{200}{549}$	$\frac{3}{122}$	$\frac{1}{45}$	$\frac{4}{45}$

				$\mathbf{15'}$	$\mathbf{15}_1$	$\mathbf{15}_2$	$\mathbf{24}$	$\mathbf{42}$
				$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
				$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$-\frac{2}{3}$	1	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{361}{2196}$	$-\frac{12}{61}$	$-\frac{1}{9}$	$\frac{5}{18}$
$\frac{1}{3}$	$\frac{1}{2}$	0	1	$-\frac{1}{3}$	$-\frac{1}{1647}$	$\frac{9}{61}$	$-\frac{4}{27}$	$\frac{10}{27}$
$\frac{1}{3}$	$\frac{3}{2}$	0	0	$\frac{3}{16}$	$\frac{25}{2928}$	$\frac{16}{61}$	$\frac{1}{3}$	$\frac{5}{24}$
$\frac{1}{3}$	$\frac{3}{2}$	0	1	$\frac{5}{48}$	$\frac{305}{432}$	0	$-\frac{5}{27}$	$\frac{1}{216}$
$\frac{4}{3}$	1	-1	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{529}{4392}$	$-\frac{24}{61}$	$\frac{2}{9}$	$\frac{5}{36}$

				$\mathbf{42}$
				$\frac{1}{3}$
				$\frac{5}{2}$
$\frac{1}{3}$	$\frac{3}{2}$	0	1	1

				$\mathbf{3}$	$\mathbf{15}_1$	$\mathbf{15}_2$	$\mathbf{42}$
				$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
				0	0	0	0
$-\frac{5}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{2}{5}$	$\frac{49}{122}$	$-\frac{6}{61}$	$-\frac{1}{10}$
$-\frac{2}{3}$	0	0	0	$-\frac{3}{20}$	$-\frac{3}{61}$	$-\frac{49}{244}$	$\frac{3}{5}$
$-\frac{2}{3}$	1	0	1	$\frac{3}{10}$	$\frac{1}{366}$	$-\frac{81}{122}$	$-\frac{1}{30}$
$\frac{1}{3}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{3}{20}$	$\frac{100}{183}$	$\frac{9}{244}$	$\frac{4}{15}$

Clebsch-Gordan Coefficients for $\mathbf{15} \otimes \mathbf{8}$

						42
						$\frac{7}{3}$
						$\frac{3}{2}$
						$-\frac{3}{2}$
$\frac{4}{3}$	1	-1	1	$\frac{1}{2}$	$-\frac{1}{2}$	1

						24	**42**
						$\frac{7}{3}$	$\frac{7}{3}$
						$\frac{1}{2}$	$\frac{3}{2}$
						$-\frac{1}{2}$	$-\frac{1}{2}$
$\frac{4}{3}$	1	-1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
$\frac{4}{3}$	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{2}{3}$

						24	**42**
						$\frac{7}{3}$	$\frac{7}{3}$
						$\frac{1}{2}$	$\frac{3}{2}$
						$\frac{1}{2}$	$\frac{1}{2}$
$\frac{4}{3}$	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
$\frac{4}{3}$	1	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$\frac{1}{3}$

						42
						$\frac{7}{3}$
						$\frac{3}{2}$
						$\frac{3}{2}$
$\frac{4}{3}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1

						15′	**42**
						$\frac{4}{3}$	$\frac{4}{3}$
						2	2
						-2	-2
$\frac{1}{3}$	$\frac{3}{2}$	$-\frac{3}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{4}{3}$	1	-1	0	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$

						15′	**15$_1$**	**15$_2$**	**24**	**42**	**42**
						$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
						2	1	1	1	1	2
						-1	-1	-1	-1	-1	-1
$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{400}{1647}$	$-\frac{1}{61}$	$-\frac{4}{27}$	$\frac{16}{27}$	0
$\frac{1}{3}$	$\frac{3}{2}$	$-\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$-\frac{529}{4392}$	$\frac{24}{61}$	$\frac{2}{9}$	$\frac{1}{72}$	$\frac{1}{8}$
$\frac{1}{3}$	$\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{8}$	$\frac{529}{13176}$	$-\frac{8}{61}$	$-\frac{2}{27}$	$-\frac{1}{216}$	$\frac{3}{8}$
$\frac{4}{3}$	1	-1	0	0	0	0	$\frac{49}{183}$	$-\frac{4}{61}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$\frac{4}{3}$	1	-1	0	1	0	$-\frac{1}{4}$	$-\frac{361}{2196}$	$-\frac{12}{61}$	$\frac{1}{9}$	$-\frac{1}{36}$	$\frac{1}{4}$
$\frac{4}{3}$	1	0	0	1	-1	$-\frac{1}{4}$	$\frac{361}{2196}$	$\frac{12}{61}$	$-\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{4}$

						$\overline{\mathbf{6}}$	**15′**	**15$_1$**	**15$_2$**	**24**	**24**	**42**	**42**
						$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
						0	2	1	1	0	1	1	2
						0	0	0	0	0	0	0	0
$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{10}$	0	$-\frac{200}{1647}$	$-\frac{1}{122}$	$\frac{2}{5}$	$-\frac{2}{27}$	$\frac{8}{27}$	0
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{10}$	0	$-\frac{200}{1647}$	$-\frac{1}{122}$	$-\frac{2}{5}$	$-\frac{2}{27}$	$\frac{8}{27}$	0
$\frac{1}{3}$	$\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$-\frac{529}{6588}$	$\frac{16}{61}$	0	$\frac{4}{27}$	$\frac{1}{108}$	$\frac{1}{4}$
$\frac{1}{3}$	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{529}{6588}$	$-\frac{16}{61}$	0	$-\frac{4}{27}$	$-\frac{1}{108}$	$\frac{1}{4}$
$\frac{4}{3}$	1	-1	0	1	1	$-\frac{4}{15}$	$-\frac{1}{12}$	$-\frac{361}{2196}$	$-\frac{12}{61}$	$-\frac{1}{15}$	$\frac{1}{9}$	$-\frac{1}{36}$	$\frac{1}{12}$
$\frac{4}{3}$	1	0	0	0	0	0	0	$\frac{49}{183}$	$-\frac{4}{61}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0
$\frac{4}{3}$	1	0	0	1	0	$\frac{4}{15}$	$-\frac{1}{3}$	0	0	$\frac{1}{15}$	0	0	$\frac{1}{3}$
$\frac{4}{3}$	1	1	0	1	-1	$-\frac{4}{15}$	$-\frac{1}{12}$	$\frac{361}{2196}$	$\frac{12}{61}$	$-\frac{1}{15}$	$-\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{12}$